Malaysian Journal of Mathematical Sciences 10(S) February: 299–304 (2016)
Special Issue: The 3rd International Conference on Mathematical Applications in Engineering 2014 (ICMAE'14)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

Dark Compactons in Nonlinear Schrödinger Lattices with Strong Nonlinearity Management

Abdul Hadi, M.S.*[1], Umarov, B.[1], Abdullaev, F.[1], and Salerno, M.[2]

[1]*Deparment of Physics, Kulliyyah of Science, International Islamic University Malaysia, 25200 Kuantan, Pahang, Malaysia*
[2]*Dipartimento di Fisica, "E.R. Caianiello", CNISM and INFN - Gruppo Collegato di Salerno, Universitá di Salerno, Via Giovanni Paolo II, 84084 Fisciano (SA), Italy*

*E-mail: salihi@iium.edu.my**

ABSTRACT

The existence of dark compacton solution of discrete nonlinear Schrödinger (DNLS) equation with strong nonlinearity management (SNLM) is investigated. The stability analysis was carry out using standard linearization stability procedure.Even though the stability regime is not so wide but evidently some stable dark compactons can exist in the SNLM DNLS system. Surprisingly, even within the confirmed stability regime from the analysis, the time evolution of the dark compacton solution exhibits small bounded ripples.

Keywords: Dark compactons, discrete nonlinear Schrödinger equation, strong nonlinearity management

1. Introduction

The study of discrete nonlinear Schrödinger equation (DNSL) perhaps was first triggered by experimental investigations in the area of nonlinear optics, particularly in fabricated AlGaAs waveguide arrays reported by Eisenberg et al. (1998) then the mathematical model was presented later in Eisenberg et al. (2000). Subsequently, the work of Morandotti et al. (2001) on the anomalous diffraction regime of AlGaAs waveguide arrays associated with Kerr-type cubic nonlinearity, has found the formation of fundamental dark soliton discrete excitations. As the DNSL is concerned, this has been known as the effect of defocusing nonlinearity. Other physical areas can be described by DNSL are Bose-Einstein condensates (BECs) and optically induced lattice (Kevrekidis, 2009).

Generation of solitons with new type of properties by periodic management of parameters of nonlinear systems has recently become attention due to its effectiveness (Malomed, 2007). The existence of discrete breathers in nonlinear lattice, in the presence nonlinear dispersion supports what is known as compactons, a nonlinear excitation without any tails (Rosenau, 1994). Even though compactons under strong nonlinearity management has been discussed in Abdullaev et al. (2010), it covers only bright compactons, so the question whether stable dark compactons can exist in the NLSM DNLS system, is open. This paper tends to investigate the dark compacton case. First, the derivation of averaged DNLSE is briefly restated. Secondly, we analyse the conditions of the existence of dark compactons and finally we investigate the linearization stability of the solution.

2. The Mathematical Model

Consider the following lattice Hamiltonian:

$$\begin{aligned} H &= -\sum_n \Big\{ \kappa \left(u_{n+1} u_n^* + u_{n+1}^* u_n \right) \\ &\quad + \frac{1}{2} \left[\gamma_0 + \gamma(t) \right] |u_n|^4 \Big\}, \end{aligned} \tag{1}$$

and it's associated dynamical system

$$\begin{aligned} i\dot{u}_n &+ \kappa \left(u_{n+1} + u_{n-1} \right) \\ &+ \left[\gamma_0 + \gamma(t) \right] |u_n|^2 u_n = 0, \end{aligned} \tag{2}$$

with the coupling constant κ quantifying the tunneling between adjacent sites, γ_0 denoting the on-site constant nonlinearity, and $\gamma(t)$ representing the time dependent modulation. It is assumed then a strong management case with $\gamma(t)$ being a periodic i.e. $\gamma(t) = \gamma(t+T)$, and rapidly varying function. A prototypical example is $\gamma(t) = \frac{\gamma_1}{\epsilon}\cos(\Omega\tau)$; with $\gamma_1 \sim O(1)$, $\epsilon \ll 1$ and $\tau = t/\epsilon$ denoting the fast time variable and $T = 2\pi/\Omega$ the period. The original system Eq.(2) can be averaged with respect to the fast time τ to yield the averaged system as follows (Abdullaev et al., 2010),

$$\begin{aligned} i\dot{u}_n &= -\alpha\kappa u_n\left[\left(u_{n+1}u_n^* + u_{n+1}^*u_n\right)J_1(\alpha\theta_+)\right. \\ &\quad \left. + \left(u_{n-1}u_n^* + u_{n-1}^*u_n\right)J_1(\alpha\theta_-)\right] \\ &\quad -\kappa\left[u_{n+1}J_0(\alpha\theta_+) + u_{n-1}J_0(\alpha\theta_-)\right] \\ &\quad -\gamma_0|u_n|^2u_n, \end{aligned} \tag{3}$$

where $\alpha = \gamma_1/\Omega$, J_i is Bessel function of order $i = 0, 1$ and $\theta_\pm = |u_{n\pm1}|^2 - |u_n|^2$.

3. Dark Compacton Modes and The Stability Analysis

The above model Eq.(3) was investigated in Abdullaev et al. (2010) for only bright compactons so the question whether compacton solutions of the NLSM DNLS system can exist on finite nonzero backgrounds is open. For stationary solutions by setting $u_n = A_n\exp(-i\mu t)$, and substitute into Eq.(3) one obtains the following equation for the real amplitudes A_n

$$\begin{aligned} \mu A_n = &-2\kappa\alpha A_n^2\left[A_{n+1}J_1(\alpha\phi_+) + A_{n-1}J_1(\alpha\phi_-)\right] \\ &-\kappa\left[A_{n+1}J_0(\alpha\phi_+) + A_{n-1}J_0(\alpha\phi_-)\right] - \gamma_0A_n^3, \end{aligned} \tag{4}$$

with $\phi_\pm = |A_{n\pm1}|^2 - |A_n|^2$. Dark modes are possible only for repulsive interactions so that in the above equation $\gamma_0 < 0$. A dark single site compacton located at the $n_0 = 0$ site is assumed by letting $A_n = b$ if $n = n_0$ and $A_n = a$ for other location of n. It follows that by fixing the chemical potential as

$$\mu = a^2\gamma_0 - 2\kappa. \tag{5}$$

one can satisfy Eq. (4) for all sites except for $n_0 \pm 1$ and n_0 for which the following equations

$$\begin{aligned} &2a\kappa J_0(\xi) + b[(a^2 - b^2)\gamma_0 - 2\kappa + 4ab\kappa\alpha J_1(\xi)] = 0, \\ &bJ_0(\xi) - a(1 + 2ab\alpha J_1(\xi)) = 0, \end{aligned} \tag{6}$$

where $\xi = \alpha\left(a^2 - b^2\right)$. This can be solved numerically to obtain $\{a, b\}$ which describes the background and the amplitude, however we observed only very small set of nontrivial solutions exist, which indicates small degree of robustness in the system with respect to stable dark compactons. In fact, one can directly check that there is no solution for $b = 0$. Fig.(1) top panel left shows a typical solution of Eq.(4) at $\kappa = 1$, based on the amplitude from Eq.(6).

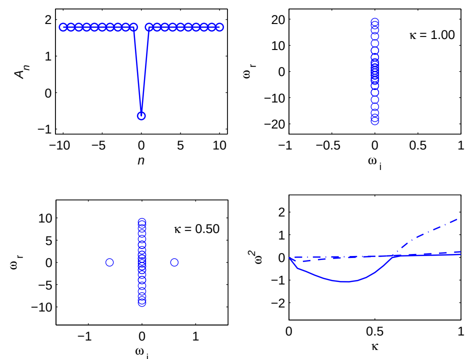

Figure 1: Top panels show the amplitude profile and eigenfrequency spectrum of one site bright-dark compactons for case $\kappa = 1$, $\gamma_0 = -1$ and $\alpha = 1$. Bottom panels show eigenfrequency spectrum for case $\kappa = 0.5$ and the numerical linear stability analysis as function of κ

Numerical stability analysis was carried out by linearizing using the ansatz:

$$u_n = \exp(-i\mu t)\left(A_n + \varepsilon\left(a_n \exp\left(-i\omega t\right) + b_n \exp\left(i\omega^* t\right)\right)\right),$$

where the ω's denote the linearization eigenfrequencies and $\varepsilon \ll 1$. Substituting the ansatz into Eq.(3) obtain the relevant (linearization) eigenvalue problem at $O\left(\varepsilon\right)$ as follows

$$\omega \begin{pmatrix} a_k \\ b_k^* \end{pmatrix} = \begin{pmatrix} \dfrac{\partial F_i}{\partial u_j} & \dfrac{\partial F_i}{\partial u_j^*} \\ \dfrac{\partial F_i^*}{\partial u_j} & \dfrac{\partial F_i^*}{\partial u_j^*} \end{pmatrix} \begin{pmatrix} a_k \\ b_k^* \end{pmatrix}$$

where

$$\begin{aligned} F_i &= \mu A_i + \gamma_0 A_i^2 A_i^* \\ &+ \kappa\left[A_{i+1} J_0\left(\phi_+\right) + A_{i-1} J_0\left(\phi_-\right)\right] \\ &+ \alpha\kappa A_i\left[\left(A_{i+1}A_i^* + A_{i+1}^* A_i\right) J_1\left(\phi_+\right)\right. \\ &+ \left.\left(A_{i-1}A_i^* + A_{i-1}^* A_i\right) J_1\left(\phi_-\right)\right]. \end{aligned}$$

The linearization stability analysis for case $\gamma_0 = -1$ and $\alpha = 1$ is depicted in right bottom panel of Fig.(1)), showing that the solutions should be stable

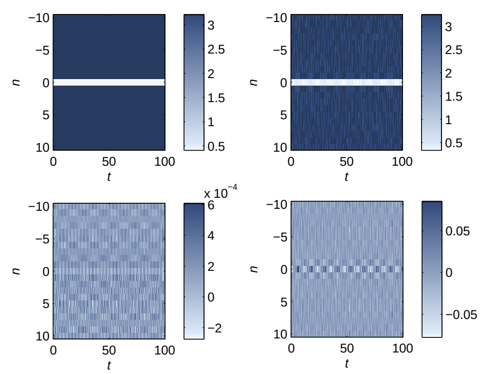

Figure 2: Space-time evolution of one site dark compacton solution for case as in top panels Fig.1. Top panels show the square modulus of the solution, the left side from system Eq.(3) and the right side from Eq.(2), while bottom panels show their respective deviation from the exact solution.

when $\kappa > 0.65$. For examples, Fig.(1) also shows the eigenfrequencies spectrum when $\kappa = 0.5, 1.0$. The existence of imaginary eigenfrequencies ω_i indicates instability for that particular case. In this case, even though the stability regime is not so wide but evidently stable dark compactons exist in the NLSM DNLS system. Furthermore, time evolutions of the dark solution from the original system and the averaged system (see Fig.(2)) are in agreement with the stability analysis. The solution for the original system however develops small bounded ripples in it's dynamics.

4. Conclusion

In this paper we have shown the existence of dark compacton under strong nonlinearity management. Despite of not being as robust as for the bright compacton case (Abdullaev et al., 2010), the strong nonlinear management of the DNLS system does provide some stable dark compacton solutions, at least from the simple ansatz, with real amplitudes we have assumed. Even though the time evolution of dark compacton develops small ripple in its dynamics, but it stays bounded thus still in agreement with the linearization stability analysis.

Acknowledgments

B. Umarov acknowledges Ministry of Higher Education, Malaysia for two-year Fundamental Research Grant Scheme under Project ID: FRGS13-092-0333.

References

Abdullaev, F. K., Kevrekidis, P. G., and Salerno, M. (2010). Compactons in nonlinear schrödinger lattices with strong nonlinearity management. *Physical Review Letters*, 105(113901):1–4.

Eisenberg, H. S., Silberberg, Y., Morandotti, R., and Aitchison, J. S. (2000). Diffraction management. *Physical Review Letters*, 85(9):1863–1866.

Eisenberg, H. S., Silberberg, Y., Morandotti, R., Boyd, A. R., and Aitchison, J. S. (1998). Discrete spatial optical solitons in waveguide arrays. *Physical Review Letters*, 81(16):3383–3386.

Kevrekidis, P. G. (2009). *The Discrete Nonlinear Schödinger Equation*. Springer-Verlag, Berlin Heidelberg.

Malomed, B. A. (2007). *Soliton Management In Periodic Systems*. Springer-Verlag, Berlin.

Morandotti, R., Eisenberg, H. S., Silberberg, Y., Sorel, M., and Aitchison, J. S. (2001). Self-focusing and defocusing in waveguide arrays. *Physical Review Letters*, 86(15):3296–3299.

Rosenau, P. (1994). Nonlinear dispersion and compact structures. *Physical Review Letters*, 73(13):1737–1741.